§10. 曲面~~的~~5运动方程.

高斯绝妙定理告诉我们曲面的第一基本型与第二基本型之间存在着某种关系. 具体并清楚得表达这种关系是这一节~~与~~~~节~~的内容。在这一节中, 我们学习利用曲面的自然标架来得到一个定义参数系上的一阶偏微分方程组; 则这阶偏微分方程组的可积性条件由Gauß-Codazzi方程表达. 此即曲面的~~基~~第I和II的关系, 称为曲面的结构方程。关于I,和II的系数

$r = r(u, v)$, 自然标架为

$$\{r; r_u, r_v, n\}, \quad 其中\ n = \frac{r_u \times r_v}{|r_u \times r_v|}$$

目标是:

$$\begin{cases} \dfrac{\partial}{\partial u}\begin{pmatrix} r \\ r_u \\ r_v \\ n \end{pmatrix} = A \begin{pmatrix} r \\ r_u \\ r_v \\ n \end{pmatrix} \\[2ex] \dfrac{\partial}{\partial v}\begin{pmatrix} r \\ r_u \\ r_v \\ n \end{pmatrix} = B \begin{pmatrix} r \\ r_u \\ r_v \\ n \end{pmatrix} \end{cases}$$

定出 A与B.

记号: $u^1 = u, \quad u^2 = v, \quad r = r(u^1, u^2)$

$$r_1 = \frac{\partial r}{\partial u^1}(u^1, u^2), \quad r_2 = \frac{\partial r}{\partial u^2}(u^1, u^2)$$

$$r_{\alpha\beta} = \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta}, \quad \alpha, \beta = 1, 2, \quad n_\alpha = \frac{\partial n}{\partial u^\alpha}, \quad \alpha = 1, 2$$

$$g_{\alpha\beta} = \langle r_\alpha, r_\beta \rangle \,, \qquad \alpha, \beta = 1, 2$$

$$b_{\alpha\beta} = \langle r_{\alpha\beta}, n \rangle = -\langle r_\alpha, n_\beta \rangle \,.$$

(注意到 $g_{\alpha\beta} = g_{\beta\alpha}$, $\quad b_{\alpha\beta} = b_{\beta\alpha}$)

Einstein 求和:

$$r_\alpha du^\alpha \triangleq \sum_{\alpha=1}^{2} r_\alpha du^\alpha = r_1 du^1 + r_2 du^2 \; (= dr)$$

$$g_{\alpha\beta} du^\alpha du^\beta = \sum_{\alpha,\beta=1}^{2} g_{\alpha\beta} du^\alpha du^\beta$$

$$= g_{11} du^1 du^1 + g_{21} du^2 du^1 + g_{12} du^1 du^2 + g_{22} du^2 du^2$$

($g_{21} du^2 du^1 + g_{12} du^1 du^2 = 2 g_{12} du^1 du^2$)

$$(= \mathrm{I})$$

$$b_{\alpha\beta} du^\alpha du^\beta = b_{11} du^1 du^1 + 2 b_{12} du^1 du^2 + b_{22} du^2 du^2$$

$$(= \mathrm{II})$$

$$(g_{\alpha\beta})^{-1} = (g^{\alpha\beta}) \,, \qquad g = \det(g_{\alpha\beta}), \quad b = \det(b_{\alpha\beta})$$

$$\Rightarrow (g^{\alpha\beta}) \cdot (g_{\alpha\beta}) = (\delta^\alpha_\beta)$$

$$\text{即} \quad g^{\alpha\beta} g_{\beta\gamma} = \delta^\alpha_\gamma \,.$$

由于 $\{r_1, r_2, n\}$ 线性无关，故

$r_{\alpha\beta}$ 与 n_α 是其线性组和. 令

$$r_{\alpha\beta} = \frac{\partial r_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta} r_\gamma + C_{\alpha\beta} n \tag{1}$$

$$\begin{cases} n_\alpha = \frac{\partial n}{\partial u^\alpha} = D^\beta_\alpha r_\beta + D_\alpha \cdot n \end{cases} \tag{2}$$

其中 $\{\Gamma^\gamma_{\alpha\beta}, C_{\alpha\beta}, D^\beta_\alpha, D_\alpha\}$ 待定系数.

(1)式与 n 作内积. 得：

$$\langle r_{\alpha\beta}, n\rangle = \Gamma^\gamma_{\alpha\beta} \underbrace{\langle r_\gamma, n\rangle}_{0} + C_{\alpha\beta} \underbrace{\langle n, n\rangle}_{1}$$

$$\| \quad b_{\alpha\beta}$$

即 $C_{\alpha\beta} = b_{\alpha\beta}$

(2)式与 n 作内积. 得

$$\underbrace{\langle n_\alpha, n\rangle}_{0} = \underbrace{\langle D^\beta_\alpha r_\beta, n\rangle}_{0} + D_\alpha \underbrace{\langle n, n\rangle}_{1}$$

$$\Rightarrow D_\alpha = 0$$

故 $$r_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} \cdot r_\gamma + b_{\alpha\beta} n \tag{3}'$$

$$\begin{cases} n_\alpha = D^\beta_\alpha r_\beta \end{cases} \tag{4}'$$

将(4)° 与 r_γ 做内积，得

$$\langle n_\alpha, r_\gamma\rangle = \langle D_\alpha^\beta r_\beta, r_\gamma\rangle$$

$$\begin{array}{ccc} \| & & \| \\ -b_{\alpha\gamma} & & D_\alpha^\beta g_{\beta\gamma} \end{array}$$

$\Rightarrow$

$$-b_{\alpha\gamma} g^{\gamma\xi} = D_\alpha^\beta g_{\beta\gamma} g^{\gamma\xi} = D_\alpha^\beta \delta_\beta^\xi = D_\alpha^\xi$$

$$\Rightarrow D_\alpha^\beta = -\left(b_{\alpha\gamma} \cdot g^{\gamma\beta}\right) \triangleq -\left(b_\alpha^\beta\right)$$ "指标的上升"

(或 $(D_\alpha^\beta) = -(b_{\alpha\beta})(g^{\alpha\beta})$

将(3)° 与 r_ξ 做内积，得

$$\langle r_{\alpha\beta}, r_\xi\rangle = \Gamma_{\alpha\beta}^\gamma g_{\gamma\xi} \qquad (5)$$

对 $g_{\alpha\beta} = \langle r_\alpha, r_\beta\rangle$ 求偏导，得

(*)

$$\frac{\partial g_{\alpha\beta}}{\partial u^\gamma} = \langle r_{\alpha\gamma}, r_\beta\rangle + \langle r_\alpha, r_{\beta\gamma}\rangle \qquad (6)$$

$$\frac{\partial g_{\alpha\gamma}}{\partial u^\beta} = \langle r_{\alpha\beta}, r_\gamma\rangle + \langle r_{\gamma\beta}, r_\alpha\rangle \qquad (7)$$

$$\frac{\partial g_{\beta\gamma}}{\partial u^\alpha} = \langle r_{\beta\alpha}, r_\gamma\rangle + \langle r_{\gamma\alpha}, r_\beta\rangle \qquad (8)$$

(6)+(7)-(8)

$$\langle r_{\beta\gamma}, r_\alpha \rangle = \frac{1}{2}\left\{ \frac{\partial g_{\alpha\beta}}{\partial u^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial u^\beta} - \frac{\partial g_{\beta\gamma}}{\partial u^\alpha} \right\}$$

或 $$\langle r_{\alpha\beta}, r_\xi \rangle = \frac{1}{2}\left\{ \frac{\partial g_{\xi\alpha}}{\partial u^\beta} + \frac{\partial g_{\xi\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi} \right\}$$

$\beta \to \alpha,\ \gamma \to \beta,\ \alpha \to \xi$

$$= \frac{1}{2}\left\{ \frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi} \right\} \qquad (9)$$

将(P)代入(5)，得

$$\Gamma^\gamma_{\alpha\beta}\, g_{\gamma\xi} = \frac{1}{2}\left\{ \frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi} \right\}$$

$$\Rightarrow \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\gamma\xi}\left\{ \frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi} \right\} \qquad (10)$$

$\{\Gamma^\gamma_{\alpha\beta}\}$ 称为曲面的 Christoffel 符号.

$$\Gamma_{\xi\alpha\beta} \overset{\Delta}{=} g_{\gamma\xi}\Gamma^\gamma_{\alpha\beta} = g_{\xi\gamma}\Gamma^\gamma_{\alpha\beta}$$

称为第二类 Christoffel 符号 "指标下降".

总结起来，我们得：

$$\begin{cases} \dfrac{\partial r}{\partial u^\alpha} = r_\alpha & \alpha=1,2 \quad (M1) \\[2ex] \dfrac{\partial r_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta} r_\gamma + b_{\alpha\beta} n & \alpha,\beta=1,2 \quad (M2) \\[2ex] \dfrac{\partial n}{\partial u^\alpha} = (-b^\beta_\alpha) r_\beta & \alpha=1,2 \quad (M3) \end{cases}$$

I + I的导数　　II　　Weingarten映射，即 $-\mathrm{II}\cdot\mathrm{I}^{-1}$

其中 $\{\Gamma^\gamma_{\alpha\beta},\ b_{\alpha\beta},\ -b^\beta_\alpha\}$ 由第 I, II 和其导数完全确定.

(M1 - M3) 称为 S 在自然标架下 $\{r; r_1, r_2, n\}$ 的运动方程.

(共有关于 2 个变量的 24 个方程组构成的一阶线性微分方程组)

考察两个变量的最简单一阶偏微分方程.

$$\begin{cases} \dfrac{\partial f}{\partial u^1} = a(u^1, u^2) \\[2ex] \dfrac{\partial f}{\partial u^2} = b(u^1, u^2) \end{cases} \qquad a, b \text{ 光滑.}$$

$\boxed{\text{这方程有解的充分必要条件是. } \dfrac{\partial a}{\partial u^2} = \dfrac{\partial b}{\partial u^1}}$:

必要性这是因为: $\dfrac{\partial^2 f}{\partial u^1 \partial u^2} = \dfrac{\partial}{\partial u^2}\left(\dfrac{\partial}{\partial u^1}\right) = \dfrac{\partial a}{\partial u^2}$

$$\| $$

$$\frac{\partial}{\partial u^1}\left(\frac{\partial f}{\partial u^2}\right)$$

$$\| $$

$$\frac{\partial b}{\partial u^1}$$

考察等式：

$$\begin{cases} \dfrac{\partial}{\partial u^\beta}\left(\dfrac{\partial r}{\partial u^\alpha}\right) = \dfrac{\partial}{\partial u^\alpha}\left(\dfrac{\partial r}{\partial u^\beta}\right) & (11) \\[2ex] \dfrac{\partial}{\partial u^\gamma}\left(\dfrac{\partial r_\alpha}{\partial u^\beta}\right) = \dfrac{\partial}{\partial u^\beta}\left(\dfrac{\partial r_\alpha}{\partial u^\gamma}\right) & (12) \\[2ex] \dfrac{\partial}{\partial u^\beta}\left(\dfrac{\partial n}{\partial u^\alpha}\right) = \dfrac{\partial}{\partial u^\alpha}\left(\dfrac{\partial n}{\partial u^\beta}\right) & (13) \end{cases}$$

将(M1)代入(11)得

$$r_{\alpha\beta} = r_{\beta\alpha}$$

$$\Leftrightarrow \Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}, \qquad b_{\alpha\beta} = b_{\beta\alpha}$$

将(M2)代入(12)得

$$\frac{\partial}{\partial u^\gamma}\left(\frac{\partial r_\alpha}{\partial u^\beta}\right) = \frac{\partial}{\partial u^\gamma}\left(\Gamma^\xi_{\alpha\beta} r_\xi + b_{\alpha\beta} n\right)$$

$$= \frac{\partial \Gamma^\xi_{\alpha\beta}}{\partial u^\gamma} r_\xi + \Gamma^\xi_{\alpha\beta} \underbrace{r_{\xi\gamma}}_{(M2)} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} n + b_{\alpha\beta} \underbrace{\frac{\partial n}{\partial u^\gamma}}_{(M3)}$$

$$= \frac{\partial \Gamma^\xi_{\alpha\beta}}{\partial u^\gamma} r_\xi + \Gamma^\xi_{\alpha\beta}\left(\Gamma^\eta_{\xi\gamma} r_\eta + b_{\xi\gamma} n\right) + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} n +$$

$$b_{\alpha\beta}\left(n - b^\xi_\gamma r_\xi\right)$$

$$= \left(\frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} - b_{\alpha\beta}b^{\xi}_{\gamma} \right) r_{\xi} + \left(\Gamma^{\xi}_{\alpha\beta} b_{\xi\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} \right) n$$

同理

$$\frac{\partial}{\partial u^{\beta}} \left(\frac{\partial r_{\alpha}}{\partial u^{\gamma}} \right)$$

$$= \left(\frac{\partial \Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta} - b_{\alpha\gamma}b^{\xi}_{\beta} \right) r_{\xi} + \left(\Gamma^{\xi}_{\alpha\gamma} b_{\xi\beta} + \frac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} \right) n.$$

$\because (r_1, r_2, \eta)$ 线性无关

Gauss 方程

$$\Rightarrow \frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta} - b_{\alpha\beta}b^{\xi}_{\gamma} + b_{\alpha\gamma}b^{\xi}_{\beta} = 0$$

$$\begin{cases} \dfrac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} - \dfrac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\xi}_{\alpha\beta} b_{\xi\gamma} - \Gamma^{\xi}_{\alpha\gamma} b_{\xi\beta} = 0 \end{cases}$$

Codazzi 方程

$\because$ 满足 (M3) 的 Γ 和 b 是

$$\frac{\partial b^{\xi}_{\beta}}{\partial u^{\gamma}} - \frac{\partial b^{\xi}_{\gamma}}{\partial u^{\beta}} = -b^{\eta}_{\beta}\Gamma^{\xi}_{\eta\gamma} + b^{\eta}_{\gamma}\Gamma^{\xi}_{\eta\beta}$$

习题：上述与 Codazzi 方程等价.

Gauss-Codazzi 方程的化简：

记号：

$$R_{\delta\alpha\beta\gamma} = g_{\delta\xi}\left(\frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta}\right)$$

Gauss 方程：

$$R_{\delta\alpha\beta\gamma} = g_{\delta\xi}(b_{\alpha\beta}b^{\xi}_{\gamma} - b_{\alpha\gamma}b^{\xi}_{\beta})$$

$$= -(b_{\alpha\gamma}b_{\delta\beta} - b_{\alpha\beta}b_{\delta\gamma})$$

断言：Gauss 方程只有一个独立方程：

$$R_{1212} = -(b_{11}b_{22} - b_{12}^2) \qquad \text{(Gauss 方程)}$$

先观察右边：(i) 若 $\alpha=\beta=\gamma=\delta$，则 右边 $\equiv 0$

(ii) 若 $\alpha,\beta,\gamma,\delta$ 中有三个相等，则右边为 0.

(iii) 只有三种情形：

$(\alpha=\beta)\neq(\gamma=\delta)$

$(\alpha=\gamma)\neq(\beta=\delta)$ 相等

$(\alpha=\delta)\neq(\beta=\gamma)$

但右式 $= \pm(b_{11}b_{22} - b_{12}^2)$

再化简左边:

回忆: $\Gamma_{\xi\alpha\beta} \triangleq g_{\xi\gamma}\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}\left\{\frac{\partial g_{\alpha\xi}}{\partial u^{\beta}} + \frac{\partial g_{\beta\xi}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\xi}}\right\}$

$$R_{\delta\alpha\beta\gamma} = \frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^{\beta}} - \frac{\partial g_{\delta\xi}}{\partial u^{\gamma}}\Gamma^{\xi}_{\alpha\beta} + \frac{\partial g_{\delta\xi}}{\partial u^{\beta}}\Gamma^{\xi}_{\alpha\gamma}$$

$$+ \Gamma^{\eta}_{\alpha\beta}\Gamma_{\delta\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\Gamma_{\delta\eta\beta}$$

$$= \frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\beta}\left(\Gamma_{\delta\eta\gamma} - \frac{\partial g_{\delta\eta}}{\partial u^{\gamma}}\right)$$

$$- \Gamma^{\eta}_{\alpha\gamma}\left(\Gamma_{\delta\eta\beta} - \frac{\partial g_{\delta\eta}}{\partial u^{\beta}}\right)$$

$$= \frac{1}{2}\left\{\cancel{\frac{\partial^2 g_{\alpha\delta}}{\partial u^{\beta}\partial u^{\gamma}}} + \frac{\partial^2 g_{\beta\delta}}{\partial u^{\alpha}\partial u^{\gamma}} - \frac{\partial^2 g_{\alpha\beta}}{\partial u^{\gamma}\partial u^{\delta}}\right.$$

$$\left. - \cancel{\frac{\partial^2 g_{\alpha\delta}}{\partial u^{\beta}\partial u^{\gamma}}} - \frac{\partial^2 g_{\gamma\delta}}{\partial u^{\alpha}\partial u^{\beta}} + \frac{\partial^2 g_{\alpha\gamma}}{\partial u^{\beta}\partial u^{\delta}}\right\}$$

$$+ \Gamma^{\eta}_{\alpha\beta}\cdot\frac{1}{2}\underbrace{\left\{-\frac{\partial g_{\delta\eta}}{\partial u^{\gamma}} + \frac{\partial g_{\delta\gamma}}{\partial u^{\eta}} - \frac{\partial g_{\eta\gamma}}{\partial u^{\delta}}\right\}}_{= -\Gamma_{\eta\gamma\delta}}$$

$$- \Gamma^{\eta}_{\alpha\gamma}\cdot\frac{1}{2}\underbrace{\left\{-\frac{\partial g_{\delta\eta}}{\partial u^{\beta}} + \frac{\partial g_{\delta\beta}}{\partial u^{\eta}} - \frac{\partial g_{\eta\beta}}{\partial u^{\delta}}\right\}}_{= -\Gamma_{\eta\beta\delta}}$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{\beta\delta}}{\partial u^\alpha \partial u^\gamma} - \frac{\partial^2 g_{\alpha\beta}}{\partial u^\gamma \partial u^\delta} - \frac{\partial^2 g_{\gamma\delta}}{\partial u^\alpha \partial u^\beta} + \frac{\partial^2 g_{\alpha\gamma}}{\partial u^\beta \partial u^\delta} \right)$$

$$- \Gamma^\eta_{\alpha\beta} \Gamma_{\eta\gamma\delta} + \Gamma^\eta_{\alpha\gamma} \Gamma_{\eta\beta\delta}$$

观察：若 $\alpha, \beta, \gamma, \delta$ 中至少三个相等，则

$$\text{左式} = 0.$$

故也只有三种情形考察.

$\beta \leftrightarrow \gamma$ 则差一负号，即 $R_{\delta\alpha\beta\gamma} = -R_{\delta\alpha\gamma\beta}$

$\beta \leftrightarrow \delta$. 则相等，即 $R_{\delta\alpha\beta\gamma} = R_{\beta\alpha\delta\gamma}$

故结论是

$$\text{左式} = \pm R_{1212}$$

故得断言.

Codazzi 方程：

$\beta = \gamma$，得 $0 = 0$，得两方位方程

$$\begin{cases} \dfrac{\partial b_{11}}{\partial u^2} - \dfrac{\partial b_{12}}{\partial u^1} + \Gamma^\xi_{11} b_{\xi 2} - \Gamma^\xi_{12} b_{\xi 1} = 0 \\ -\dfrac{\partial b_{22}}{\partial u^1} + \dfrac{\partial b_{21}}{\partial u^2} - \Gamma^\xi_{22} b_{\xi 1} + \Gamma^\xi_{21} b_{\xi 2} = 0 \end{cases}$$

$$\begin{pmatrix} b_{22} & -b_{21} \\ -b_{12} & b_{11} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial u^1} \\ \frac{\partial}{\partial u^2} \end{pmatrix} = -\begin{pmatrix} \Gamma_{22}^{\beta} & -\Gamma_{21}^{\beta} \\ -\Gamma_{12}^{\beta} & \Gamma_{11}^{\beta} \end{pmatrix}\begin{pmatrix} b_{1\beta} \\ b_{2\beta} \end{pmatrix}$$ (Codazzi 方程)

注意：由 Gauss 方程，我们还得到高斯曲率公式：

$$K = \frac{\det \mathrm{II}}{\det \mathrm{I}} = \frac{b}{g} = \frac{-R_{1212}}{g}$$

或 $gK = -R_{1212}$。

总结：

曲面 $\longrightarrow$ 运动方程，基本微分形式 $\Rightarrow$ Gauss-Codazzi 方程

特别地：Gauss 方程 $\Rightarrow$ Gauss 绝妙定理。

改写：

运动方程 + Gauss-Codazzi 方程 $\longrightarrow$ 曲面，

设 $D = \{(u^1, u^2)\} \subset \mathbb{R}^2$ 上有：

$$\varphi = g_{\alpha\beta} du^\alpha du^\beta; \quad \Psi = b_{\alpha\beta} du^\alpha du^\beta$$

它们上有二次微分形式，且 $(g_{\alpha\beta})$，$(b_{\alpha\beta})$ 对称，且 $(g_{\alpha\beta}) > 0$

通过记 $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1}$，$b_\alpha^\beta = g^{\beta\gamma} b_{\gamma\alpha}$.

$$\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\delta}\left\{\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\beta\delta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right\}$$

定义：若 $\Gamma_{\alpha\beta}^{\gamma}$, $b_{\alpha\beta}$, b_α^β 满足 Gauss - Codazzi 方程，则

$\forall (u_0^1, u_0^2) \in D$，存在 u_0 的邻域 $U \subset D$，以及定义在 U 上的曲面 $r(u_1, u_2): U \to \mathbb{R}^3$，使得 $\Gamma_{\alpha\beta}^{\gamma}$, $b_{\alpha\beta}$, b_α^β 为曲面 r 的基本型。

证明：考虑以 r, r_1, r_2, n 为未知函数的偏微分方程组

$$\begin{cases} \dfrac{\partial r}{\partial u^\alpha} = r_\alpha \\ \dfrac{\partial r_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^{\gamma} r_\gamma + b_{\alpha\beta} n \\ \dfrac{\partial n}{\partial u^\alpha} = -b_\alpha^\beta r_\beta \end{cases} \qquad (*)$$

由于 Gauss - Codazzi 方程为上述偏微分方程组可积的条件.

故对给定点 $u_0 \in D$，及 r^0, r_α^0, n^0（易另外给出）

有 u_0 的邻域 U 以及定义在 U 上的函数 $\{r, r_2, n\}$ 满足(*)

及初值条件 $\begin{cases} r(u_0^1, u_0^2) = r^0 \\ r_\alpha(u_0^1, u_0^2) = r_\alpha^0, \quad \alpha = 1, 2 \\ n(u_0^1, u_0^2) = n^0 \end{cases}$

现取 r^0, r^0_α, n^0 满足

$$\begin{cases} \langle r^0_\alpha, r^0_\beta \rangle = g_{\alpha\beta}(u_0) \\ \langle r^0_\alpha, n^0 \rangle = 0 \\ \langle n^0, n^0 \rangle = 1, \quad (r^0_1, r^0_2, n^0) > 0 \end{cases} \qquad (**)$$

以下证明方程(*)的解满足初值(**)的解 r 为所求曲面.

即 r 为曲面且其 I, II 为给定 φ, ψ:

对此, 我们考虑由 u 上的函数 $\{\langle r_\alpha, r_\beta \rangle, \langle r_\alpha, n \rangle, \langle n, n \rangle\}$

由(*)推出它们满足以下一阶线性偏微分方程组

$$\begin{cases} \dfrac{\partial \langle r_\alpha, r_\beta \rangle}{\partial n^\gamma} = \Gamma^\xi_{\alpha\gamma} \langle r_\xi, r_\beta \rangle + \Gamma^\xi_{\beta\gamma} \langle r_\xi, r_\alpha \rangle \\ \qquad + b_{\alpha\gamma} \langle r_\beta, n \rangle + b_{\beta\gamma} \langle r_\alpha, n \rangle \\ \dfrac{\partial \langle r_\alpha, n \rangle}{\partial n^\beta} = \Gamma^\xi_{\alpha\beta} \langle r_\xi, n \rangle + b_{\alpha\beta} \langle n, n \rangle - b^\gamma_\beta \langle r_\alpha, r_\gamma \rangle \\ \dfrac{\partial \langle n, n \rangle}{\partial n^\alpha} = -2 b^\beta_\alpha \langle r_\beta, n \rangle \end{cases} \qquad (***)$$

显然, $\langle r_\alpha, r_\beta \rangle = g_{\alpha\beta}$, $\langle r_\alpha, n \rangle = 0$, $\langle n, n \rangle = 1$ 满足(***).

根据一阶偏微分方程组关于初值问题解的唯一性知.

$\langle r_\alpha, r_\beta \rangle$ 必等于 $g_{\alpha\beta}$, $\langle r_\alpha, n \rangle = 0$ 及 $\langle n, n \rangle = 1$

故 r_1, r_2 线性无关，且单位向量 n 与 r_1, r_2 垂直.

由于 $(r_1, r_2, n)(u_0) > 0$，故 $\{r_1, r_2, n\}$ 在 u_0 点保持正号，故

$$n = \frac{r_1 \times r_2}{|r_1 \times r_2|}$$

故再由 (*) 的第二式 $\equiv 0$

$$\text{II} = \langle r_{\alpha\beta}, n \rangle du^\alpha du^\beta = b_{\alpha\beta} du^\alpha du^\beta = 0.$$

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定理：设 S_i, $i=1,2$ 为定义在 $D \subset \mathbb{R}^2$ 上的两个曲面. 若 $\forall (u,v) \in D$，$r_1(u,v)$ 与 $r_2(u,v)$ 的第一、第二基本形式分别相等，即 $\text{I}_1 = \text{I}_2$, $\text{II}_1 = \text{II}_2$，则 $\exists\, T \in SO(3) \ltimes \mathbb{R}^3$，使得 $r_2 = T \circ r_1$.

证明：习题.

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